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Manifolds and Group actions

Homework 6

Suggested Exercise 1.
Consider the Lie group $GL_n(\mathbb{R})$ and let $[\cdot,\cdot]$ denote the Lie bracket. Show that

$$[A,B] = AB - BA. \tag{1}$$

Mandatory Exercise 1. (10 Points)
Let M be a submanifold of N. A vector field X on N is said to be tangent to M if for all for all $p \in M$ the vector $X(p) \in T_pM \subset T_pN$. Note that a vector field tangent to M can be restricted to M.

a) Let X, Y be vector fields on N, tangent to M. Show that $[X, Y]$ is also a vector field tangent to M and that its restriction to M coincides with the Lie bracket of the vector fields X and Y when restricted to M.

b) A matrix group is a Lie subgroup of $GL_n(\mathbb{R})$. Show that formula (1) also holds for matrix groups.

Mandatory Exercise 2. (5 Points)
Consider the Lie group

$$SU(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) \mid AA^* = \mathrm{id}, \quad \text{and} \quad \det A = 1 \right\},$$

where A^* denotes the conjugate transpose of A.

1. Show that $SU(2)$ is equal to

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, \quad \text{with} \quad |a|^2 + |b|^2 = 1 \right\},$$

   and conclude that $SU(2)$ is diffeomorphic to S^3.

2. Show that the matrices

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

   are a basis of the lie algebra of $SU(2)$. Compute their Lie brackets.
   Remark: one can actually show that the lie algebras of $SU(2)$ and $SO(3)$ are isomorphic. If you feel like it you can try to construct an isomorphism. As manifolds $SU(2)$ and $SO(3)$ are not diffeomorphic: $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$ and $SU(2)$ is diffeomorphic to S^3, the universal cover of $SO(3)$.

Mandatory Exercise 3. (5 Points)
Let G be a matrix group. Show that the identity

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)} Y = [X, Y]$$

holds for all $X \in \mathfrak{g} := \mathrm{Lie}(G)$.

Hand in on Monday 29 of May in the pigeonhole on the third floor.